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Abstract

We give a direct tensor decomposition for any density matrix into Hermitian operators. Based upon the decomposition we study when the mixed states are separable and generalize the separability indicators to multi-partite states and show that a density operator is separable if and only if the separable indicator is non-negative. We then derive two bounds for the separable indicator in terms of the spectrum of the factor operators in the tensor summands.

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1. Introduction

In the last decade quantum entanglement has played a remarkable role in many applications and become one of the key resources in the rapidly expanding fields of quantum information and quantum computation, especially in quantum teleportation, quantum cryptography, quantum dense coding and parallel computation [1–3]. A quantum state or density matrix is separable (or not entangled) if it is a convex sum of tensor product of quantum states. In this case the separable quantum state can be prepared in several different locations. There are two aspects in the question regarding quantum entanglement: the first is to judge whether a general quantum state is entangled or not, and the second is to establish how much entanglement remains after some noisy quantum process. In the case of pure states, the Bell inequality provides a useful tool to tell separability from entanglement [4]. In [5–8] the separability problem was examined and important criteria were proposed from several viewpoints for the far more difficult case of mixed states including the PPT criterion and the range equality condition. In terms of measurement of entanglement other methods have been found, e.g. formation of entanglement [9] and purification of formation [11, 12]. Recently further important and interesting works [13–15] have also been devoted solely to quantum entanglement and some criteria were proposed accordingly, in particular, [16] gives an operational and geometric

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approach to pairwise entanglement of two and three-dimensional composite quantum systems. Despite these important developments the question of separability still remains unsolved and is notoriously famous for its difficulty.

Among the approaches to quantum separability it is important to have an operational method to decompose the quantum states as a tensor product. Such an idea was first studied in [17], where some necessary constraints were found to ensure an optimal separable approximation to a given density matrix, and then a numerical method is proposed to locate the optimal separable state for two-partite mixed states. In [18] a new algebraic mechanism was introduced to study the separability question for two partite mixed states. The idea was first to decompose the mixed density matrix as a summation of tensor products of Hermitian operators, and then we rearrange the sum to get the indicator. It was proved that the density matrix is separable if and only if the indicator is non-negative. Thus the indicator provides a new measurement for the separability.

In this paper we will generalize this method to multi-partite density operators. We will give a new operational method to decompose the density matrix as a summation of tensor products of Hermitian operators. Our new method at the simplest case is the fundamental fact that any 4×4 Hermitian operator is a span of composite Pauli spin matrices $\sigma_i \otimes \sigma_j$, where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \quad \text{and} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Once the decomposition into tensor products is known, the idea of the separability indicator [18] is generalized into multi-partite states and we show that the mixed states are separable if and only if the separability indicator is non-negative. In general it is hard to compute the separability indicator. For this purpose we provide several bounds, and hope that they will help in the determination of the separability.

2. Basic notions

Let H_1 (resp. H_2) be an m (resp. n)-dimensional complex Hilbert space, with $|i\rangle, i = 1, 2 \ldots, m$ (resp. $|j\rangle, j = 1, 2 \ldots, n$) as an orthonormal basis. A bipartite mixed state is said to be separable if the density matrix can be written as

$$\rho = \sum_i p_i \rho_i^1 \otimes \rho_i^2, \tag{1}$$

where $0 < p_i \leqslant 1$, $\sum_i p_i = 1$, ρ_i^1 and ρ_i^2 are density matrices on H_1 and H_2 respectively. It is a challenging problem to find such a decomposition or to prove that it does not exist for a generic mixed state [5–8].

We first introduce some notations. For an $m \times m$ block matrix Z with each block Z_{ij} of size $n \times n$, $i, j = 1, 2 \ldots, m$. The realigned matrix $\tilde{Z}$ is defined by

$$\tilde{Z} = [\mathrm{vec}(Z_{11}), \ldots, \mathrm{vec}(Z_{m1}), \ldots, \mathrm{vec}(Z_{1m}), \ldots, \mathrm{vec}(Z_{mm})]^t, \tag{2}$$

where for any $m \times n$ matrix T with entries t_{ij}, $\mathrm{vec}(T)$ is defined to be

$$\mathrm{vec}(T) = [t_{11}, \ldots, t_{m1}, t_{12}, \ldots, t_{1n}, \ldots, t_{mn}]^t.$$

Let $A = A^R + \sqrt{-1} A^I$ be a complex Hermitian matrix, where A^R and A^I are real and imaginary parts of A. Let σ be the canonical map from A to a real matrix:

$$\sigma : A \longmapsto \begin{pmatrix} A^R & A^I \\ -A^I & A^R \end{pmatrix}, \tag{3}$$

where A^R and A^I are the real and imaginary parts of A respectively.

Let Q_s be an $m^2 \times \frac{m(m-1)}{2}$ matrix. If we arrange the row indices of Q_s as

$$\{11, 21, 31, \ldots, m1, 12, 22, 32, \ldots, m2, \ldots, mm\},$$

then all the entries of Q_s are zero except those at 21 and 12 (resp. 31 and 13, . . .) which are 1 and -1 respectively in the first (resp. second, . . .) column. In other words,

$$Q_s = [\{e_{21}, -e_{12}\}; \{e_{31}, -e_{13}\}; \ldots; \{e_{m,m-1}, -e_{m-1,m}\}],$$

where $\{e_{21}, -e_{12}\}$ is first column of Q_s, with 1 and -1 at the 21 and 12 rows respectively; while $\{e_{31}, -e_{13}\}$ is second column of Q_s, with 1 and -1 at the 31 and 13 rows respectively; and so on.

Let Q_a be an $m^2 \times \frac{m(m+1)}{2}$ matrix such that

$$Q_a = [\{e_{11}\}; \{e_{21}, e_{12}\}; \{e_{31}, e_{13}\}, \ldots; \{e_{22}\}; \{e_{32}, e_{23}\},$$
$$\{e_{42}, e_{24}\}; \ldots; \{e_{m,m-1}, e_{m-1,m}\}, \{e_{mm}\}],$$

where $\{e_{11}\}$ is the column vector with 1 at the row ii and zero elsewhere, and $\{e_{1j}, e_{1j}\}$ is the column vector with 1 at the ijth and jith rows and zero elsewhere. Q_1 can be expressed as

$$Q_1 = \begin{pmatrix} \overline{Q_s} & 0 & 0 & \overline{Q_a} \\ 0 & \overline{Q_a} & \overline{Q_s} & 0 \end{pmatrix},$$

where $\overline{Q_s}$ and $\overline{Q_a}$ are obtained by normalizing each column of Q_s and Q_a.

By replacing the dimension m with n, we have Q_2.

As an example we have for $m = 2$

$$Q_s = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \qquad Q_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3. The tensor product decompositions of Hermitian matrices

Let A be a Hermitian matrix on Hilbert space $H_1 \otimes H_2$. In [18] we gave an operational method to decompose A as a tensor product of Hermitian matrices on H_1 and H_2. We will give another method to decompose A and then generalize to the case of multi-tensor products.

Let us recall the decomposition method in [18]. We express the matrix A in terms of real and complex parts: $A = A^R + iA^I$ and realign both A^R and A^I into $\tilde{A}^R$ and $\tilde{A}^I$ respectively as in equation (2). Then we write

$$Q_1^t \begin{pmatrix} \tilde{A}^R & \tilde{A}^I \\ -\tilde{A}^I & \tilde{A}^R \end{pmatrix} Q_2 = \begin{pmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{pmatrix}. \tag{4}$$

Proposition 1. *Let A be an $mn \times mn$ Hermitian matrix as rewritten in equation (4). Suppose the singular value decomposition of $\hat{A}_{22}$ is $\hat{A}_{22} = \sum_{i=1}^{r} \sqrt{\lambda_i} u_i v_i^t$, where r is the rank and λ_i ($i = 1, 2, \ldots r$) are the non-zero eigenvalues of $\hat{A}_{22}^{\dagger} \hat{A}_{22}$, and u_i (resp. v_i) are the eigenvectors of the matrix $\hat{A}_{22} \hat{A}_{22}^{\dagger}$ (resp. $\hat{A}_{22}^{\dagger} \hat{A}_{22}$). Set $\hat{B}_i = \sqrt{\lambda_i} u_i$, $\check{C}_i = -v_i$. Then we can decompose A as a tensor product*

$$A = \sum_{i=1}^{r} B_i \otimes C_i,$$

where the $m \times m$ Hermitian matrices $B_i = b_i + \sqrt{-1}\mathcal{B}_i$ and the $n \times n$ Hermitian matrices $C_i = c_i + \sqrt{-1}\mathcal{C}_i$ are given by

$$\begin{pmatrix} \text{vec}(b_i) \\ -\text{vec}(\mathcal{B}_i) \end{pmatrix} = Q_1 \begin{pmatrix} 0 \\ -\hat{\mathcal{B}}_i \end{pmatrix}, \qquad \begin{pmatrix} \text{vec}(c_i) \\ \text{vec}(\mathcal{C}_i) \end{pmatrix} = Q_2 \begin{pmatrix} 0 \\ \check{\mathcal{C}}_i \end{pmatrix}. \tag{5}$$

The above result gives a constructive or operative method to decompose A as a tensor product. The existence of tensor decomposition has a simpler explanation. In fact, we know that the set of $n \times n$ Hermitian matrices is a real vector space of dimension n^2, thus the dimension of Hermitian matrices of size $mn \times mn$ is exactly equal to the product of the dimension of size $m \times m$ and that of size $m \times m$, hence the subspace of tensor product of Hermitian matrices of size $n \times n$ and that of size $m \times m$ must equal to the space of all Hermitian matrices of size $mn \times mn$, which guarantees the existence.

We observe that in general the space of real symmetric (antisymmetric) matrices cannot be decomposed into a tensor product of symmetric (antisymmetric) matrices. In fact, the difference between dimensions of the space of $mn \times mn$ symmetric matrices and that of the tensor product of symmetric matrices of size $m \times m$ and size $n \times n$ is

$$\binom{mn+1}{2} - \binom{m+1}{2}\binom{n+1}{2} = \binom{m}{2}\binom{n}{2}.$$

Similarly the difference between the dimensions of antisymmetric operators over $\mathbb{C}^m \times \mathbb{C}^n$ and that of the tensor product of antisymmetric operators is

$$\binom{mn-1}{2} - \binom{m-1}{2}\binom{n-1}{2} = \binom{m+1}{2}\binom{n+1}{2} - 1.$$

We can use induction to generalize proposition 1 to multi-partite case.

Theorem 1. *Let A be an Hermitian matrix on space $H_1 \otimes H_2 \otimes H_3 \otimes \ldots \otimes H_n$. A has tensor production decomposition like $A = \sum_{i=1}^{r} B_i^1 \otimes B_i^2 \otimes \ldots \otimes B_i^n$, where $B_i^1, B_i^2, \ldots, B_i^n$ are Hermitian matrices on $H_1, H_2, \ldots, H_n$ respectively.*

We now present a practical method to decompose Hermitian matrices into a tensor product of Hermitian matrices, thus giving a new constructive proof for theorem 1. Let E_{ij}^n be the unit square matrices of size $n \times n$. If it is clear form the context, we will omit the superscript. To decompose the unit matrix E_{ij}^{mn}, we write its indices i, j uniquely as follows:

$$i = (k-1)n + i', \qquad j = (l-1)n + j', \tag{6}$$

where $1 \leqslant k, l \leqslant m$ and $1 \leqslant i', j' \leqslant n$. Then we have

$$E_{ij}^{mn} = E_{kl}^m \otimes E_{i'j'}^n. \tag{7}$$

Equivalently we can picture the above decomposition as follows. We first view E_{ij}^{mn} as an $m \times m$ block matrix with each entry as an $n \times n$ matrix. The resulted block matrix is still a unit-like matrix where all entries are zero except (k, l)-entry, which is an $n \times n$ unity matrix itself, say $E_{i'j'}$. Then we immediately have $E_{ij}^{mn} = E_{kl}^m \otimes E_{i'j'}^n$.

Example 1. Let $(1 + 7b)^{-1}\rho_b$ be the density operator on $\mathbb{C}^2 \otimes \mathbb{C}^4$ as follows.

$$\rho_b = \begin{pmatrix} b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & b \\ 0 & 0 & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1+b}{2} & 0 & 0 & \frac{\sqrt{1-b^2}}{2} \\ b & 0 & 0 & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & b & 0 & \frac{\sqrt{1-b^2}}{2} & 0 & 0 & \frac{1+b}{2} \end{pmatrix}.$$

We can decompose ρ_b by the above scheme.

$$
\begin{aligned}
\rho_b &= b\left(E_{11}^8 + E_{16}^8 + E_{22}^8 + E_{27}^8 + E_{33}^8 + E_{38}^8 + E_{44}^8 + E_{61}^8 + E_{66}^8 + E_{72}^8 + E_{77}^8 + E_{83}^8\right) \\
&\quad + \frac{1+b}{2}\left(E_{55}^8 + E_{88}^8\right) + \frac{\sqrt{1-b^2}}{2}\left(E_{58}^8 + E_{85}^8\right) \\
&= b\big(E_{11}^2 \otimes E_{11}^4 + E_{12}^2 \otimes E_{12}^4 + E_{11}^2 \otimes E_{22}^4 + E_{12}^2 \otimes E_{23}^4 + E_{11}^2 \otimes E_{33}^4 + E_{12}^2 \otimes E_{34}^4 \\
&\quad + E_{11}^2 \otimes E_{44}^4 + E_{21}^2 \otimes E_{21}^4 + E_{22}^2 \otimes E_{22}^4 + E_{21}^2 \otimes E_{32}^4 + E_{22}^2 \otimes E_{33}^4 + E_{21}^2 \otimes E_{43}^4\big) \\
&\quad + \frac{1+b}{2}\left(E_{22}^2 \otimes E_{11}^4 + E_{22}^2 \otimes E_{44}^4\right) + \frac{\sqrt{1-b^2}}{2}\left(E_{22}^2 \otimes E_{14}^4 + E_{22}^2 \otimes E_{41}^4\right).
\end{aligned}
$$

For a different decomposition using the singular value decomposition, the reader is referred to [18].

This decomposition method can be generalized to Hermitian operators. Let A be a Hermitian matrix, then one can decompose A into a sum of real and imaginary parts: $A = B + \sqrt{-1}C$, where B (or C) is a symmetric (or antisymmetric) matrix. Let $\{E_{ij} + E_{ji}\}$ be the basis for the symmetric matrices, and $\{E_{ij} - E_{ji}\}$ be the basis for the antisymmetric matrices. It is enough to decompose the basis elements as tensor products of Hermitian matrices. Roughly speaking, one writes each basis element $E_{ij} \pm E_{ji}$ of size $mn \times mn$ as a block matrix, then transform it into a tensor product according to the position where the 1 or -1 appears. The main point is that we have to consider all Hermitian matrices to factor the basis elements (cf the remark after proposition 1).

Specifically, by modulo n we write the indices i, j uniquely as in equation (6): $i \equiv i' (\mathrm{mod}\, n)$, $j \equiv j' (\mathrm{mod}\, n)$ and $k = [(i - 1)/n] + 1$, $l = [(j - 1)/n] + 1$. Here the representatives for $\mathbb{Z}_n$ are taken to be $\{1, 2, \dots, n\}$. Then we have the decomposition

$$
E_{ij}^{mn} + E_{ji}^{mn} = \tfrac{1}{2}\left[\left(E_{kl}^m + E_{lk}^m\right) \otimes \left(E_{i'j'}^n + E_{j'i'}^n\right) - \sqrt{-1}\left(E_{kl}^m - E_{lk}^m\right) \otimes \sqrt{-1}\left(E_{i'j'}^n - E_{j'i'}^n\right)\right],
$$

(8)

$$
\begin{aligned}
\sqrt{-1}\left(E_{ij}^{mn} - E_{ji}^{mn}\right) = \tfrac{1}{2}\Big[&\left(E_{kl}^m + E_{lk}^m\right) \otimes \sqrt{-1}\left(E_{i'j'}^n - E_{j'i'}^n\right) \\
&+ \sqrt{-1}\left(E_{kl}^m - E_{lk}^m\right) \otimes \left(E_{i'j'}^n + E_{j'i'}^n\right)\Big].
\end{aligned}
$$

(9)

Equivalently we can picture the above decomposition as follows. We first view $E_{ij}^{mn} \pm E_{ji}^{mn}$ as an $m \times m$ block matrix (P_{st}), where $P_{st} = 0$ except $(s, t) = (k, l)$ or (l, k), and $P_{kl} = P_{lk}^T = E_{i'j'}$. Then we have $E_{ij}^{mn} + E_{ji}^{mn} = E_{kl}^m \otimes E_{i'j'}^n + E_{lk}^m \otimes E_{j'i'}^n$. A simple computation will show that it is also given by equation (8).

Example 2. For $f \in [0, 1]$ consider the Werner state [19]

$$\rho = \begin{pmatrix} \frac{1-f}{3} & & & \\ & \frac{1+2f}{6} & \frac{1-4f}{6} & \\ & \frac{1-4f}{6} & \frac{1+2f}{6} & \\ & & & \frac{1-f}{3} \end{pmatrix}. \tag{10}$$

Then

$$\begin{aligned} \rho &= \frac{1-f}{3} E_{11}^4 + \frac{1+2f}{6} \left(E_{22}^4 + E_{33}^4\right) + \frac{1-4f}{6} \left(E_{23}^4 + E_{32}^4\right) + \frac{1-f}{3} E_{44}^4 \\ &= \frac{1-f}{3} E_{11} \otimes E_{11} + \frac{1+2f}{6} (E_{11} \otimes E_{22} + E_{22} \otimes E_{11}) \\ &\quad + \frac{1-4f}{12} [(E_{12} + E_{21}) \otimes (E_{21} + E_{12}) - \mathrm{i}(E_{12} - E_{21}) \otimes \mathrm{i}(E_{21} - E_{12})] + \frac{1-f}{3} E_{22} \otimes E_{22}. \end{aligned}$$

Example 3. For non-negative a, b, c consider the following positive semi-definite matrix:

$$\rho = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{a} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{b} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{c} & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

Then we have

$$\begin{aligned} \rho &= E_{11} \otimes E_{11} \otimes E_{11} + E_{22} \otimes E_{22} \otimes E_{22} \\ &\quad + \frac{1}{4} S_{12} \otimes (S_{12} \otimes S_{12} - \mathrm{i}A_{12} \otimes \mathrm{i}A_{12}) - \frac{1}{4}\mathrm{i}A_{12} \otimes (S_{12} \otimes \mathrm{i}A_{12} - \mathrm{i}A_{12} \otimes S_{12}) \\ &\quad + aE_{11} \otimes E_{11} \otimes E_{22} + bE_{11} \otimes E_{22} \otimes E_{11} + cE_{11} \otimes E_{22} \otimes E_{22} \\ &\quad + \frac{1}{a} E_{22} \otimes E_{11} \otimes E_{11} + \frac{1}{b} E_{22} \otimes E_{11} \otimes E_{22} + \frac{1}{c} E_{22} \otimes E_{22} \otimes E_{11}, \end{aligned}$$

where $S_{ij} = E_{ij} + E_{ji}$ and $A_{ij} = E_{ij} - E_{ji}$.

4. Separability of multi-partite states

As we note in the previous section that any Hermitian operator A on a tensor product space can be decomposed into a sum of tensor products of Hermitian operators: $A = \sum_{i=1}^{r} B_i^1 \otimes B_i^2 \otimes \cdots \otimes B_i^n$. However the factors B_i^j are generally not density matrices on H_j as they may not be positive operators. To answer the question of separability of A one needs to study when each factor is non-negative.

Let $m(A)$ and $M(A)$ denote the smallest and the largest eigenvalues of a Hermitian matrix A. We can transform the decomposition into another one so that the smallest eigenvalues are non-negative as follows:

$$A = \sum_{i=1}^{r} B_i^1 \otimes B_i^2 \otimes \cdots \otimes B_i^n$$

$$= \sum_{i=1}^{r} \left(B_i^1 - m\left(B_i^1\right) \mathrm{Id}_1 + m\left(B_i^1\right) \mathrm{Id}_1 \right) \otimes \cdots \otimes \left(B_i^n - m\left(B_i^n\right) \mathrm{Id}_n + m\left(B_i^n\right) \mathrm{Id}_n \right)$$

$$= \sum_{i=1}^{t} B_i'^1 \otimes B_i'^2 \otimes \cdots \otimes B_i'^n + q(A) \, \mathrm{Id}_1 \otimes \mathrm{Id}_2 \otimes \cdots \otimes \mathrm{Id}_n, \tag{12}$$

where $B_i'^j$ are positive semi-definite Hermitian matrices on H_j, and each summand has at least one $m\left(B_k'^j\right) = 0$ but not all (i.e. at least one factor is the identity Id_l on H_l).

Note that $q(A)$ depends on the decomposition. We define the *separability indicator* of A, $S(A) = \max(q(A))$ to be the maximum value of $q(A)$ among all possible decompositions such as (12). The following result is quoted from [18].

Proposition 2. *Let $A = \sum_i^r B_i \otimes C_i$ be a density matrix on space $H_1 \otimes H_2$. Then A is separable iff the separability indicator $S(A) \geqslant 0$. Moreover $S(A)$ satisfies the following relation $S(A) \leqslant m(A)$.*

Theorem 2. *Let $A = \sum_{i=1}^{r} B_i \otimes C_i$ be a Hermitian operator on the space $H_1 \otimes H_2$, then $q(A)$ is given by*

$$q(A) = \sum_{m(B_i) \geqslant 0, m(C_i) \geqslant 0} m(B_i) m(C_i) + \sum_{m(B_i) < 0} m(B_i) M(C_i)$$
$$+ \sum_{m(C_i) < 0} M(B_i) m(C_i) - \sum_{m(B_i) < 0, m(C_i) < 0} m(B_i) m(C_i), \tag{13}$$

and bounded by

$$q(A) \geqslant M(A) - \sum_{i=1}^{r} [(M(B_i) - m(B_i))(M(C_i) - m(C_i)) + M(m(C_i) B_i)$$
$$- m(m(C_i) B_i) + M(m(B_i) C_i) - m(m(B_i) C_i)]. \tag{14}$$

Proof. For any Hermitian matrix P we define the operation P' by shifting with the minimum eigenvalue: $P' = P - m(P)I$. We can rewrite equation (12)

$$A = \sum_{i=1}^{r} B_i' \otimes C_i' + (m(C_i) B_i' - m((m(C_i) B_i') I_m) \otimes I_n$$
$$+ I_m \otimes (m(B_i) C_i' - m(m(B_i) C_i') I_n) + q(A) I_m \otimes I_n,$$

where

$$q(A) = \sum_i (m(m(C_i) B_i') + m(m(B_i) C_i') + m(B_i) m(C_i)).$$

We observe that for any real s,

$$m(sP) = \frac{s + |s|}{2} m(P) + \frac{s - |s|}{2} M(P), \qquad M(sP) = \frac{s + |s|}{2} M(P) + \frac{s - |s|}{2} m(P). \tag{15}$$

It then follows that

$$q(A) = \sum_i (m(m(C_i)B_i') + m(m(B_i)C_i') + m(B_i)m(C_i))$$

$$= \sum_{m(B_i) \geqslant 0, m(C_i) \geqslant 0} m(B_i)m(C_i)$$

$$+ \sum_{m(B_i) < 0, m(C_i) \geqslant 0} m(B_i)M(C_i) + \sum_{m(B_i) \geqslant 0, m(C_i) < 0} M(B_i)m(C_i)$$

$$+ \sum_{m(B_i) < 0, m(C_i) < 0} (m(C_i)M(B_i) + m(B_i)M(C_i) - m(B_i)m(C_i))$$

$$= \sum_{m(B_i) \geqslant 0, m(C_i) \geqslant 0} m(B_i)m(C_i) + \sum_{m(B_i) < 0} m(B_i)M(C_i)$$

$$+ \sum_{m(C_i) < 0} M(B_i)m(C_i) - \sum_{m(B_i) < 0, m(C_i) < 0} m(B_i)m(C_i).$$

Now we note that for any matrix P and any real number r, $M(P - rI) = M(P) - r$, $m(P - rI) = m(P) - r$, from which it follows that

$$M(B_i') = M(B_i - m(B_i)I_m) = M(B_i) - m(B_i), \tag{16}$$

$$M(m(C_i)B_i' - m(m(C_i)B_i')I_m) = M(m(C_i)B_i) - m(m(C_i)B_i).$$

On the other hand it is well known that $M(A + B) \leqslant M(A) + M(B)$ (see [20]). Thus taking the maximum eigenvalues, we get

$$q(A) \geqslant M(A) - \sum_{i=1}^{r} [(M(B_i) - m(B_i))(M(C_i) - m(C_i)) + M(m(C_i)B_i)$$

$$- m(m(C_i)B_i) + M(m(B_i)C_i) - m(m(B_i)C_i)],$$

which completes the proof.   □

In the last part of the proof if we take minimum eigenvalues we will get the known inequality $m(A) \geqslant q(A)$ (using $m(A + B) \geqslant m(A) + m(B)$).

We remark that the above lower bound is different from that in [18]. To better understand our lower bounds, we consider the special case when all factors are non-negative matrices, then $m(m(B_i)C_i) = m(B_i)m(C_i)$ etc. Then it follows that

$$q(A) \geqslant M(A) - \sum_{i=1}^{r} [M(B_i)M(C_i) - m(B_i)m(C_i)]. \tag{17}$$

While the other extreme case is when all factors are negative, then

$$q(A) \geqslant M(A) - \sum_{i=1}^{r} [(M(B_i) - 2m(B_i))(M(C_i) - 2m(C_i)) - m(B_i)m(C_i)].$$

When the factors B_i or C_i are not all non-negative, we have

$$q(A) \geqslant M(A) - \sum_{m(B_i) \geqslant 0, m(C_i) \geqslant 0} [M(B_i)M(C_i) - m(B_i)m(C_i)]$$

$$- \sum_{m(B_i) < 0, m(C_i) \geqslant 0} [(M(B_i) - 2m(B_i))M(C_i) + m(B_i)m(C_i)]$$

$$- \sum_{m(B_i)\geqslant 0, m(C_i)<0} [M(B_i)(M(C_i) - 2m(C_i)) + m(B_i)m(C_i)]$$

$$- \sum_{m(B_i)<0, m(C_i)<0} [(M(B_i) - 2m(B_i))(M(C_i) - 2m(C_i)) - m(B_i)m(C_i)].$$

$$(18)$$

The above result can be generalized to multi-partite states.

Theorem 3. *Let $A = \sum_i^r B_i \otimes C_i \otimes D_i$ be a density matrix on space $H_1 \otimes H_2 \otimes H_3$. Then A is separable if and only if the separability indicator $S(A) \geqslant 0$. Moreover $S(A)$ satisfies the following relations:*

$$S(A) \leqslant m(A). \tag{19}$$

$$q(A) \geqslant M(A) - \sum_{i=1}^r [(M(B_i) - m(B_i))(M(C_i) - m(C_i))(M(D_i) - m(D_i))$$

$$+ M(m(B_i)m(D_i)C_i) - m(m(B_i)m(D_i)C_i) + M(m(C_i)m(D_i)B_i)$$

$$- m(m(C_i)m(D_i)B_i) + M(m(B_i)m(C_i)D_i) - m(m(B_i)m(C_i)D_i)$$

$$+ M(m(m(B_i)C_i)D_i - m(B_i)m(C_i)D_i) - m(m(m(B_i)C_i)D_i - m(B_i)m(C_i)D_i)$$

$$+ M(m(m(C_i)B_i)D_i - m(C_i)m(B_i)D_i) - m(m(m(C_i)B_i)D_i - m(C_i)m(B_i)D_i)$$

$$+ M(m(m(D_i)B_i)C_i - m(D_i)m(B_i)C_i) - m(m(m(D_i)B_i)C_i - m(D_i)m(B_i)C_i)$$

$$+ (M(m(D_i)B_i) - m(m(D_i)B_i))(M(C_i) - m(C_i))$$

$$+ (M(m(C_i)B_i) - m(m(C_i)B_i))(M(D_i) - m(D_i))$$

$$+ (M(m(B_i)C_i) - m(m(B_i)C_i))(M(D_i) - m(D_i))]. \tag{20}$$

The idea of the proof will be similar to that of theorem 2 and is included in the appendix. More generally we can use the same idea to give similar results for multi-partite cases.

Theorem 4. *Let A be a k-partite mixed state on space $H_1 \otimes H_2 \otimes \cdots \otimes H_k$, then A has a tensor decomposition into Hermitian operators in the form $A = \sum_{i=1}^r B_i^1 \otimes B_i^2 \otimes \cdots \otimes B_i^k$ and is separable if and only if the separability indicator $S(A) \geqslant 0$. Moreover $S(A)$ satisfies the following relation:*

$$S(A) \leqslant m(A). \tag{21}$$

When all factors are non-negative, we have

$$q(A) = \sum_i^r m(B_i^1)m(B_i^2) \cdots m(B_i^k)$$

$$\geqslant M(A) - \sum_{i=1}^r [M(B_i^1)M(B_i^2) \cdots M(B_i^k) - m(B_i^1)m(B_i^2) \cdots m(B_i^k)]. \tag{22}$$

5. Conclusion

We have developed a criterion to judge whether a multi-partite density operator is separable. Our idea is first to decompose the density operator into a sum of tensor product of Hermitian operators. We give a new and practical way to decompose any Hermitian operator into tensor product of Hermitian operators in multi-partite cases. Unlike the numerical method [17] and

the method of singular value decomposition [18] our new method is completely elementary and algebraic. Using the decomposition we can rewrite it into a tensor product of positive operators plus a scalar operator, which is called the separability indicator. The separability indicator provides a new mechanism to measure the quantum entanglement of the density operator. We derive some bound to estimate the scalar or separability indicator. Our inequalities are expressed in terms of eigenvalues of the summands, and in some case they are sufficient to tell if the separability indicator is non-negative, thus shows that the density operator is separable. As our method relies on how the operator is decomposed, it is usually difficult to compute the separability indicator exactly. We hope our estimates will shed more light on the separability problem.

Acknowledgments

Appendix

Proof of theorem 3. The idea of the proof is similar to that of theorem 2. Recall the meaning of operation $P' = P - m(P)I$, and we have

$$A = \sum_{i=1}^{r} B_i \otimes C_i \otimes D_i = \sum_{i=1}^{r} B_i' \otimes C_i' \otimes D_i' + \sum_{i=1}^{r} (m(D_i)B_i' - m(m(D_i)B_i')I_m) \otimes C_i' \otimes I_k$$

$$+ \sum_{i=1}^{r} I_m \otimes (m(m(D_i)B_i')C_i' - m(m(m(D_i)B_i')C_i')I_n) \otimes I_k$$

$$+ \sum_{i=1}^{r} I_m \otimes [m(B_i)m(D_i)C_i' - m(m(B_i)m(D_i)C_i')I_n] \otimes I_k$$

$$+ \sum_{i=1}^{r} I_m \otimes (m(B_i)C_i' - m(m(B_i)C_i')I_n) \otimes D_i'$$

$$+ \sum_{i=1}^{r} (m(C_i)B_i' - m(m(C_i)B_i')I_m) \otimes I_n \otimes D_i'$$

$$+ \sum_{i=1}^{r} I_m \otimes I_n \otimes (m(m(B_i)C_i')D_i' - m(m(m(B_i)C_i')D_i')I_k)$$

$$+ \sum_{i=1}^{r} [m(C_i)m(D_i)B_i' - m(m(C_i)m(D_i)B_i')I_m] \otimes I_n \otimes I_k$$

$$+ \sum_{i=1}^{r} I_m \otimes I_n \otimes (m(m(C_i)B_i')D_i' - m(m(m(C_i)B_i')D_i')I_k)$$

$$+ \sum_{i=1}^{r} I_m \otimes I_n \otimes [m(B_i)m(C_i)D_i' - m(m(B_i)m(C_i)D_i')I_k]$$

$$+ q(A)I_m \otimes I_n \otimes I_k, \tag{A.1}$$

where

$$q(A) = \sum_{i=1}^{r} m(m(m(D_i)B_i')C_i') + \sum_{i=1}^{r} m(m(m(B_i)C_i')D_i') + \sum_{i=1}^{r} m(m(m(C_i)B_i')D_i')$$

$$+ \sum_{i=1}^{r} m(m(C_i)m(D_i)B_i') + \sum_{i=1}^{r} m(m(B_i)m(D_i)C_i') + \sum_{i=1}^{r} m(m(B_i)m(C_i)D_i')$$

$$+ \sum_{i=1}^{r} m(B_i)m(C_i)m(D_i)$$

$$= \sum_{i=1}^{r} m(m(m(D_i)B_i)C_i - m(D_i)m(B_i)C_i)$$

$$+ \sum_{i=1}^{r} m(m(m(B_i)C_i)D_i - m(B_i)m(C_i)D_i)$$

$$+ \sum_{i=1}^{r} m(m(m(C_i)B_i)D_i - m(B_i)m(C_i)D_i) - \sum_{i=1}^{r} m(m(D_i)B_i)m(C_i)$$

$$- \sum_{i=1}^{r} m(m(B_i)C_i)m(D_i) - \sum_{i=1}^{r} m(m(C_i)B_i)m(D_i) + \sum_{i=1}^{r} m(B_i)m(C_i)m(D_i),$$

$$(A.2)$$

where we have used similar identities like equation (16). Now we would like to consider eight possible signs of $m(B_i), m(C_i), m(D_i)$, and we use $+, -, +$ to denote the subset $\{i | m(B_i) \geqslant 0, m(C_i) < 0, m(D_i) \geqslant 0\}$ etc. to simplify the notation. By equation (15) it follows that

$$q(A) = \sum_{+,+,+} m(B_i)m(C_i)m(D_i)$$

$$+ \sum_{-,+,+} m(B_i)[M(C_i)M(D_i) - M(C_i)m(D_i) - m(C_i)M(D_i) - m(C_i)m(D_i)]$$

$$+ \sum_{+,-,+} m(C_i)[M(B_i)M(D_i) - m(B_i)M(D_i) - M(B_i)m(D_i) - m(B_i)m(D_i)]$$

$$+ \sum_{+,+,-} m(D_i)[M(B_i)M(C_i) - m(B_i)M(C_i) - M(B_i)m(C_i) - m(B_i)m(C_i)]$$

$$+ \sum_{-,-,+} [m(B_i)(M(C_i)M(D_i) - M(C_i)m(D_i) - m(C_i)M(D_i))$$

$$+ m(C_i)(M(B_i)M(D_i) - m(B_i)M(D_i) - M(B_i)m(D_i))]$$

$$+ \sum_{-,+,-} [m(B_i)(M(C_i)M(D_i) - M(C_i)m(D_i) - m(C_i)M(D_i))$$

$$+ m(D_i)(M(B_i)M(C_i) - m(B_i)M(C_i) - M(B_i)m(C_i))]$$

$$+ \sum_{+,-,-} [m(C_i)(M(B_i)M(D_i) - m(B_i)M(D_i) - M(B_i)m(D_i))$$

$$+ m(D_i)(M(B_i)M(C_i) - m(B_i)M(C_i) - M(B_i)m(C_i))]$$

$$+ \sum_{-,-,-} (m(D_i)M(C_i)M(B_i) + m(B_i)M(C_i)M(D_i) + m(C_i)M(D_i)M(B_i)$$

$$- 2m(B_i)m(C_i)M(D_i) - 2m(B_i)m(D_i)M(C_i) - 2m(C_i)m(D_i)M(B_i)$$

$$+ m(B_i)m(C_i)m(D_i)).$$

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The above expression leads to an easy proof of the criterion: if A is separable, then all the factors are non-negative and $S(A) \geqslant q(A) = \sum_i m(B_i)m(C_i)m(D_i) \geqslant 0$. The converse is immediate.

If we take minimum eigenvalues to the decomposition (12), we will get

$$m(A) \geqslant \sum_i m\left(B_i'^1\right)m\left(B_i'^2\right)\cdots m\left(B_i'^m\right) + q(A) = q(A). \tag{A.3}$$

Next using similar identities as equation (16) we get identities like

$$M(m(D_i)B_i' - m(m(D_i)B_i')I_m)) = M(m(D_i)B_i) - m(m(D_i)B_i),$$
$$M(m(m(D_i)B_i')C_i' - m(m(m(D_i)B_i')C_i')I_n) = M(m(m(D_i)B_i)C_i - m(D_i)m(B_i)C_i)$$
$$- m(m(m(D_i)B_i)C_i - m(D_i)m(B_i)C_i).$$

Thus we have

$$\begin{aligned}
M(A) \leqslant \sum_{i=1}^{r} &[(M(B_i) - m(B_i))(M(C_i) - m(C_i))(M(D_i) - m(D_i)) \\
&+ M(m(B_i)m(D_i)C_i) - m(m(B_i)m(D_i)C_i) + M(m(C_i)m(D_i)B_i) \\
&- m(m(C_i)m(D_i)B_i) + M(m(B_i)m(C_i)D_i) - m(m(B_i)m(C_i)D_i) \\
&+ M(m(m(B_i)C_i)D_i - m(B_i)m(C_i)D_i) - m(m(m(B_i)C_i)D_i - m(B_i)m(C_i)D_i) \\
&+ M(m(m(C_i)B_i)D_i - m(C_i)m(B_i)D_i) - m(m(m(C_i)B_i)D_i - m(C_i)m(B_i)D_i) \\
&+ M(m(m(D_i)B_i)C_i - m(D_i)m(B_i)C_i) - m(m(m(D_i)B_i)C_i - m(D_i)m(B_i)C_i) \\
&+ (M(m(D_i)B_i) - m(m(D_i)B_i))(M(C_i) - m(C_i)) \\
&+ (M(m(C_i)B_i) - m(m(C_i)B_i))(M(D_i) - m(D_i)) \\
&+ (M(m(B_i)C_i) - m(m(B_i)C_i))(M(D_i) - m(D_i))] + q(A),
\end{aligned} \tag{A.4}$$

which completes the proof of theorem 3. □

We remark that when all factors are non-negative matrices, then it follows that

$$q(A) \geqslant M(A) - \sum_{i=1}^{r}[M(B_i)M(C_i)M(D_i) - m(B_i)m(C_i)m(D_i)]. \tag{A.5}$$

While the other extreme case is when all factors are negative, then

$$\begin{aligned}
q(A) \geqslant M(A) - \sum_{i=1}^{r} &[(M(B_i) - 3m(B_i))(M(C_i) \\
&- 3m(C_i))(M(D_i) - 3m(D_i)) - m(B_i)m(C_i)m(D_i)].
\end{aligned}$$

Proof of theorem 4. The proof is by an easy induction as those of theorems 2 and 3. Some details are already offered in equations (A.3) and (A.4). □

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